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Solitons and discrete eigenfunctions of the recursion operator of non-linear evolution equations: I. The Caudrey–Dodd–Gibbon–Sawada–Kotera equation

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Abstract. The solution of the third-order isospectral equation of the Caudrey-Dodd-Gibbon-Sawada-Kotera equation (CDGSKE) for soliton potential is obtained recursively from the Riccati equation derived by iterating once the auto-Bäcklund transformation. It is then shown that the discrete eigenfunctions of the sixth-order recursion operator for this equation can be written in terms of the solutions of the isospectral equation. The behaviour of the 1-soliton solution which has certain novel features is studied. A sine-Gordon-like equation resembling the double-sine-Gordon equation is derived from the CDGSKE.

1. Introduction

Most of the known integrable non-linear evolution equations (NLEE) have the following common features.

(a) Associated with such an equation is an isospectral linear eigenvalue problem.

(b) For every solution there exists an infinity of constants of motion in involution. These NLEE can be described as a Hamiltonian system with the constants of motion of the Hamiltonian. Thus with a given integrable NLEE is associated an infinite hierarchy of integrable equations.

(c) About every solution there exists an infinity of symmetries or infinitesimal transformations (IT). This infinity of IT can be generated either through an integro-differential recursion operator for NLEE in one space dimension or through a Lie product (Fokas and Fuchssteiner 1981a, Oevel and Fuchssteiner 1982, Aiyer 1984a). These IT also generate the infinite hierarchy of integrable NLEE mentioned in (b) above.

(d) These equations possess an auto-Bäcklund transformation (ABT) which connects two solutions of the equation.

(e) The spatial part of the ABT and the isospectral equation are the same for every member of a hierarchy of integrable NLEE.

The fifth-order Caudrey-Dodd-Gibbon-Sawada-Kotera equation (Sawada and Kotera 1974, Dodd and Gibbon 1977) (CDGSKE) given by

$$u_t + u_{5x} + 30uu_{3x} + 30u_xu_{xx} + 180u^2u_x = 0 \tag{1.1}$$

while sharing all the features mentioned above has one important difference. The recursion operator generating the infinity of IT about a solution $u(x, t)$ of (1.1) does

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not connect the IT $u_x(x, t)$ to the IT $u_t(x, t)$ (Fuchssteiner and Oevel 1982) whereas for the KdVE, modified KdVE, sine-Gordon equation and the non-linear Schrödinger equation the recursion operator connects the IT $u_x(x, t)$ and $u_t(x, t)$ (Fuchssteiner 1979, Aiyer 1983a). The Hirota-Satsuma coupled KdVE is similar to the CDGSKE in this respect (Fuchssteiner 1982, Aiyer 1983b). As a consequence there exist about every solution of the CDGSKE two distinct infinite sets of IT. This we believe gives rise to other differences namely, the ABT for the CDGSKE is a differential equation of the second order and the isospectral equation is of the third order (Dodd and Gibbon 1977, Satsuma and Kaup 1977). This point will be explained later. For the other NLEE mentioned above (except the Hirota-Satsuma equation) the ABT is a first-order differential equation and the isospectral equation is a pair of coupled first-order differential equations.

We have studied some aspects of the CDGSKE and the results are summarised below. Details are given in the subsequent sections. For convenience we discuss the results in terms of the dependent variable $w(x, t)$ related to $u(x, t)$, a solution of (1.1), by

$$u(x, t) = w_x(x, t). \tag{1.2}$$

The evolution equation for $w(x, t)$ is then

$$w_t + w_{5x} + 30 w_x w_{3x} + 60 w_x^3 = 0. \tag{1.3}$$

Our notation is as follows. A 1-soliton solution of (1.3) with parameter k_i is denoted by $w_i(x, t)$ or sometimes simply as w_i. The parameter k_i determines the velocity of the soliton. A 2-soliton with parameters k_1, k_2 is denoted by $w_{1,2}$. Generally an n-soliton with parameters $k_1, k_2, \ldots, k_n$ is denoted by $w_{1,n}$. These parameters determine the velocities of the n 1-solitons as $t \to \infty$. We emphasise that $w_n(x, t)$ in our notation is a 1-soliton with parameter k_n and not an n-soliton.

2. Summary of results

We summarise our results as follows.

(a) Explicit solutions of the third-order isospectral transform with an n-soliton $w_{1,n}(x, t)$ as potential are constructed recursively. If $k_1, k_2, \ldots, k_n$ are the n parameters of $w_{1,n}(x, t)$ we show that $2n$ discrete eigenfunctions, two for each k_i, of the sixth-order recursion operator $T(w_{1,n})$ of the CDGSKE can be written in terms of the solutions of the third-order isospectral transform. We thus have explicitly $2n$ discrete eigenfunctions of the sixth-order recursion operator.

(b) The time behaviour of the 1-soliton solution of the CDGSKE has a novel feature. The general 1-soliton solution has been derived and numerically evaluated for two special cases.

(c) The sine-Gordon equation (SGE) can be obtained from the KdVE in the following way:

(i) KdV → MKdV: by the well known Miura transform (Miura 1968);

(ii) MKdV → potential MKdV: by transforming the dependent variable $\mathcal{V}(x, t)$ in MKdV to $D^{-1}\mathcal{V}(x, t)$;

(iii) potential MKdV → SG: by applying the inverse of the recursion operator (Fuchssteiner 1979, Aiyer 1983a, Fokas and Fuchssteiner 1981b). We similarly obtain a modified CDGSK and the SG-like equation from the CDGSKE. The interesting feature of this equation is its resemblance to the double SGE (DSGE) (Bullough *et al* 1980).

Now we turn to some elaboration. By an n-soliton solution $w_{1,n}(x, t)$ of (1.3) we mean the solution $\bar{w}(x, t)$ obtained by n applications of the ABT (Dodd and Gibbon 1977, Satsuma and Kaup 1977)

$$(\bar{w}-w)_{xx}+(\bar{w}-w)^3+3(\bar{w}-w)(\bar{w}+w)_x=k^3 \tag{2.1}$$

starting from the solution $w=0$ of (1.3). It can be shown that if w (or $\bar{w}$) is a solution of (1.3) then $\bar{w}$ (or w) satisfying (2.1) is also a solution of (1.3).

If in (2.1) $w=0$ and $k=k_1$ then $\bar{w}$ is the 1-soliton w_1 with parameter k_1. In the next step if $w=w_1$ and $k=k_2$ in (2.1) then $\bar{w}$ is the 2-soliton $w_{1,2}$ with parameters k_1, k_2. Such successive application of (2.1) gives $\bar{w}=w_{1,n}$ an n-soliton with parameters $k_1, k_2, \ldots, k_n$ (see the earlier section on notation).

With

$$\bar{w}-w=\psi_x/\psi \tag{2.2}$$

equation (2.1) can be put in the form

$$\psi_{3x}+6w_x\psi_x=k^3\psi \tag{2.3}$$

which is the isospectral equation associated with (1.3) (Dodd and Gibbon 1977, Satsuma and Kaup 1977). Let us consider an infinitesimal change $y(x, t)$ of $w(x, t)$ and denote the corresponding change in $\bar{w}(x, t)$ by $z(x, t)$. Thus

$$w\to w+\varepsilon y \qquad \bar{w}\to\bar{w}+\varepsilon z.$$

Substituting these in (2.1) and retaining terms linear in ε we obtain

$$\begin{aligned} z_{xx}&+3(\bar{w}-w)^2z+3(\bar{w}-w)z_x+3(\bar{w}+w)_xz \\ &=y_{xx}+3(\bar{w}-w)^2-3(\bar{w}-w)y_x+3(\bar{w}+w)_xy \end{aligned} \tag{2.4}$$

which with $y=0$ reduces to

$$z_{xx}+3(\bar{w}-w)z_x+3[(\bar{w}-w)^2+(\bar{w}+w)_x]z=0 \tag{2.5}$$

i.e. a linear second-order equation in z. This means that there are two linearly independent IT about $\bar{w}(x, t)$ corresponding to the zero IT ($y=0$) about $w(x, t)$. This we think arises from the two independent sets of IT about a solution of (1.3). The ABT (2.1) and consequently (2.5) being of second order is not surprising for this alone would lead to two linearly independent IT about $\bar{w}(x, t)$. This also explains why the isospectral equation (2.3) is of third order. Later we will show that the solutions of (2.5) can be written in terms of the solution of (2.3) and that this is of third order will again give rise to two linearly independent IT which are solutions of (2.5).

We obtain an interesting result for the non-linear superposition principle from the ABT (2.1). Starting from a known solution $w(x, t)$ of (1.3) we represent by $W_i(x, t)$ the solution obtained by adding one soliton with parameter k_i to $w(x, t)$. By $W_{ji}(x, t)(W_{ij}(x, t))$ we represent the solution obtained by adding one soliton with parameter $k_i(k_j)$ to $W_j(W_i)$. Note that we are using the notation W_i, W_j, W_{ij} and *not* w_i, w_j or w_{ij}. The latter are 1- or 2-soliton solutions obtained from the ABT by one or two iterations starting from $w=0$. $W_i(W_j)$ and W_{ij} on the other hand are obtained from the ABT by one or two iterations starting from any solution w of (1.3), not necessarily equal to zero. Assuming that the Bianchi permutability holds we have $W_{ji}(x, t)=W_{ij}(x, t)$. This equality leads to a first-order differential equation in $W_{ij}(x, t)$. We do not obtain the algebraic superposition principle as in the case of the KdVE or SGE for, in these cases, to start with, the ABT was a first-order differential equation.

One could go a step further, that is derive an algebraic superposition principle for $W_{ijk}(x, t)$, the solution being obtained by adding a soliton to $W_{ij}(x, t)$. But the first-order differential equation for $W_{ij}(x, t)$ is a simple Riccati equation which can be easily solved. The very interesting and useful result that follows from the solution of this Riccati equation is that we can solve the scattering equation (2.3) with an n-soliton as potential (that is $w = w_{1,n}$ in (2.3)) in terms of the solution of (2.3) with an $(n-1)$-soliton as potential, up to a quadrature.

This in turn enables us to obtain explicitly the solutions of (2.5) since the solutions of (2.5) can be written in terms of the solutions of (2.3). Then again the solutions of (2.5) (if $\bar{w} = w_{1,n}$ and $w = w_{1,n-1}$ are soliton solutions) are eigenfunctions of the sixth-order recursion operator of the CDGSKE. We thus have explicitly the discrete eigenfunctions of the recursion operator.

The 1-soliton solution of (1.3) also presents certain novel features. From (2.1)-(2.3) it follows that the 1-soliton $w_1(x, t)$ is of the form

$$w_1(x, t) = \psi_x(x, t)/\psi(x, t) \tag{2.6}$$

where

$$\psi_{3x} = k_1^3\psi. \tag{2.7}$$

The general solution of (2.7) such that (2.6) satisfies (1.3) is

$$\psi = [A_1 \exp -\mathrm{i}\sqrt{3}(\beta x + at) + A_2 \exp(\beta x - at) + A_3 \exp -(\beta x - at)] \exp(\mathrm{i}\beta/\sqrt{3})x \tag{2.8}$$

where

$$\beta = (\mathrm{i}\sqrt{3}/2)k_1 \qquad a = 16\beta^4. \tag{2.9}$$

a can be found by substituting (2.6) in (1.3). The time dependence of $w_1(x, t)$ is the most interesting feature of this result. The time dependences of the oscillatory and non-oscillatory parts are different. The solution given by (2.6) and (2.8) changes shape with time with a definite period. The usual solution (Dodd and Gibbon 1977)

$$w_1(x, t) \sim \beta \tanh(\beta x - at) \tag{2.10}$$

is obtained from (2.6) and (2.8) with $A_1 = 0$ and A_2, A_3 arbitrary but of the same sign. As a consequence of this time dependence (2.8)

$$w_{1t} \neq \text{constant} \times w_{1x}. \tag{2.11}$$

This can be traced to the fact that in the 1-soliton manifold obtained by putting $w = 0$ in (2.1) (or given by (2.6) and (2.7)) the spatial derivative part in the CDGSKE (1.3), namely $w_{5x} + 30w_x w_{3x} + 60w_x^3$, does not reduce to constant $\times\, w_x$ but to constant $\times$ $(w_{xx} + 2ww_x)$.

The details are covered in the following sections. In § 3 the first-order Riccati equation for $W_{ij}(x, t)$ is derived from the ABT and solved. In § 4 the solution of the isospectral equation with n-solitons as potential is constructed recursively. In § 5 the solutions of (2.5) are obtained in terms of the solutions of the isospectral equation (2.3) and it is then shown that the solutions of (2.5) are eigenfunctions of the sixth-order recursion operator. In § 6 the general 1-soliton solution is derived and its behaviour discussed. The two discrete independent IT about the 1-soliton are discussed. In § 7 the SG-like equation obtained from the CDGSKE is derived and its similarity to the DSGE displayed.

3. The derivation and solution of the Riccati equation obtained by iterating the ABT

As before let $\bar{w} = W_1(x, t)(\bar{w} = W_2(x, t))$ be the solution of (1.3) obtained from the ABT (2.1) by adding one soliton with parameter $k_1(k_2)$ to $w(x, t)$ where $w(x, t)$ is any solution of (1.3). Assuming that the Bianchi permutability holds we represent by the common symbol $W_{1,2}(x, t)$ the solution obtained from the ABT by adding a soliton with parameter $k_2(k_1)$ to $W_1(x, t)(W_2(x, t))$. Then from (2.1) follow the equations

$$(W_1 - w)_{xx} + (W_1 - w)^3 + 3(W_1 - w)(W_1 + w)_x = k_1^3 \qquad (3.1a)$$

$$(W_2 - w)_{xx} + (W_2 - w)^3 + 3(W_2 - w)(W_2 + w)_x = k_2^3 \qquad (3.1b)$$

$$(W_{1,2} - W_1)_{xx} + (W_{1,2} - W_1)^3 + 3(W_{1,2} - W_1)(W_{1,2} + W_1)_x = k_2^3 \qquad (3.1c)$$

$$(W_{1,2} - W_2)_{xx} + (W_{1,2} - W_2)^3 + 3(W_{1,2} - W_2)(W_{1,2} + W_2)_x = k_1^3. \qquad (3.1d)$$

Subtracting (3.1d) from (3.1c) the resulting first-order equation in $W_{1,2}$ is

$$\begin{aligned}(W_2 - W_1)_{xx} &+ (W_2 - W_1)(3W_{1,2}^2 - 3W_1W_{1,2} - 3W_2W_{1,2} + W_1^2 + W_2^2 + W_1W_2)\\ &+ 3W_{1,2}(W_1 - W_2)_x + 3(W_2 - W_1)(W_{1,2})_x + 3W_2W_{2x} - 3W_1W_{1x}\\ &= k_2^3 - k_1^3. \end{aligned} \qquad (3.2)$$

From (3.1a, b) follows the equation

$$\begin{aligned}(W_2 - W_1)_{xx} &+ (W_2 - W_1)(3w^2 - 3wW_1 - 3wW_2 + W_1^2 + W_2^2 + W_1W_2)\\ &+ 3w(W_1 - W_2)_x + 3W_2W_{2x} - 3W_1W_{1x} + 3W_2(W_2 - W_1)\\ &= k_2^3 - k_1^3. \end{aligned} \qquad (3.3)$$

Substituting for $(W_2 - W_1)_{xx}$ in (3.2) from (3.3) and simplifying one obtains

$$\begin{aligned}(W_{1,2} - w)_x[(W_2 - w) - (W_1 - w)] &+ (W_{1,2} - w)^2[(W_2 - w) - (W_1 - w)]\\ &- (W_{1,2} - w)\{(W_2 - w)^2 - (W_1 - w)^2 + [(W_2 - w)_x - (W_1 - w)_x]\} = 0.\end{aligned}$$

This can be written as

$$(\tilde{W}_{1,2})_x + \tilde{W}_{1,2}^2 - \tilde{W}_{1,2}[\tilde{W}_2 + \tilde{W}_1 + (\tilde{W}_2 - \tilde{W}_1)_x/(\tilde{W}_2 - \tilde{W}_1)] = 0 \qquad (3.4)$$

where

$$\begin{aligned}\tilde{W}_{1,2} &= W_{1,2} - w\\ \tilde{W}_1 &= W_1 - w\\ \tilde{W}_2 &= W_2 - w.\end{aligned} \qquad (3.5)$$

This is a Riccati equation and is linearised by the substitution

$$\tilde{W}_{1,2} = \Psi_x/\Psi. \qquad (3.6)$$

Equation (3.4) reduces to

$$\Psi_{xx} - \Psi_x[\tilde{W}_2 + \tilde{W}_1 + (\tilde{W}_2 - \tilde{W}_1)_x/(\tilde{W}_2 - \tilde{W}_1)] = 0. \qquad (3.7)$$

If $\tilde{W}_1$, $\tilde{W}_2$ is written in the form

$$\begin{aligned}\tilde{W}_1 &= f_{1x}/f_1\\ \tilde{W}_2 &= f_{2x}/f_2\end{aligned} \qquad (3.8)$$

one then immediately obtains

$$\Psi_x = f_{2x} f_1 - f_2 f_{1x} \tag{3.9}$$

so that obtaining $\tilde{W}_{1,2}$ is reduced to the integration of (3.9).

4. Solutions of the isospectral equation (2.3) with soliton as potential

Using the results of the last section we construct the solutions of

$$\psi_{3x} + 6(w_{1,n})_x \psi_x = k^3 \psi \tag{4.1}$$

for an n-soliton $w_{1,n}$ with parameters $k_1, \ldots, k_n$ as the potential.

In § 3 we considered solutions $\bar{w}(x, t) = W_1(x, t)$ (or $W_2(x, t)$) of the ABT (2.1) starting from an arbitrary solution $w(x, t)$ of (1.3). We now restrict $w(x, t)$ to be a soliton solution. If in particular, we consider w to be an $(n-2)$-soliton with parameters $k_3, k_4, \ldots, k_n$, then $W_1 = \bar{w}$ obtained from (2.1) with $k = k_1$ is an $(n-1)$-soliton with parameters $k_1, k_3, \ldots, k_n$. Similarly $W_2 = \bar{w}$ with $k = k_2$ is also an $(n-1)$-soliton with parameters $k_2, k_3, \ldots, k_n$. The solution $W_{1,2}$ of § 3 is then an n-soliton with parameters $k_1, k_2, \ldots, k_n$, that is $w_{1,n}$.

In this section we will therefore denote these w, W_1, W_2 and $W_{1,2}$ by

$$w = w_{1,n-2}(k_1, k_2) \tag{4.2a}$$

the subscript indicating that it is an $(n-2)$-soliton and (k_1, k_2) indicating that the $(n-2)$ parameters of this $(n-2)$-soliton do not contain k_1, k_2 from the sequence $k_1, k_2, k_3, \ldots, k_n$. Similarly

$$W_1 = w_{1,n-1}(k_2) \tag{4.2b}$$

indicates that it is an $(n-1)$-soliton which does not contain the parameter k_2. Then

$$W_2 = w_{1,n-1}(k_1) \tag{4.2c}$$

and

$$W_{1,2} = w_{1,n}. \tag{4.2d}$$

Further we can write

$$w_{1,n-2}(k_1, k_2) = F_x / F.$$

To distinguish that F is associated with $w_{1,n-2}(k_1, k_2)$ we write

$$F \equiv \psi_{1,n-2}(k_1, k_2)$$

so that

$$w_{1,n-2}(k_1, k_2) = [\psi_{1,n-2}(k_1, k_2)]_x / \psi_{1,n-2}(k_1, k_2). \tag{4.2e}$$

From (3.5), (3.6) and (4.2) we have

$$\begin{aligned} w_{1,n} &= \Psi_x / \Psi + [\psi_{1,n-2}(k_1, k_2)]_x / \psi_{1,n-2}(k_1, k_2) \\ &= [\Psi \psi_{1,n-2}(k_1, k_2)]_x / [\Psi \psi_{1,n-2}(k_1, k_2)]. \end{aligned} \tag{4.3}$$

Similarly from (3.5), (3.8) and (4.2)

$$w_{1,n-1}(k_1) = [f_2 \psi_{1,n-2}(k_1, k_2)]_x / f_2 \psi_{1,n-2}(k_1, k_2). \tag{4.4}$$

On the other hand if

$$w_{1,n} - w_{1,n-1}(k_1) = g_x(k_1)/g(k_1) \tag{4.5}$$

then from (2.1)-(2.3) and (3.1*d*)

$$(g(k_1))_{3x} + 6(w_{1,n-1}(k_1))_x(g(k_1))_x = k_1^3 g(k_1). \tag{4.6}$$

From (4.3)-(4.5) we have

$$\frac{\partial}{\partial x}\ln g(k_1) = \frac{\partial}{\partial x}\ln\{\Psi\psi_{1,n-2}(k_1, k_2)\} - \frac{\partial}{\partial x}\ln\{f_2\psi_{1,n-2}(k_1, k_2)\}$$

$$= \frac{\partial}{\partial x}\ln\{\Psi/f_2\} \tag{4.7}$$

or

$$g(k_1) = \Psi/f_2. \tag{4.8}$$

Since from (3.9)

$$\Psi = D^{-1}(f_{2x}f_1 - f_2 f_{1x})$$

we have

$$g(k_1) = D^{-1}(f_{2x}f_1 - f_2 f_{1x})/f_2. \tag{4.9}$$

Now from (3.5) and (3.8)

$$w_{1,n-1}(k_1) - w_{1,n-2}(k_1, k_2) = f_{2x}/f_2 \tag{4.10a}$$

$$w_{1,n-1}(k_2) - w_{1,n-2}(k_1, k_2) = f_{1x}/f_1. \tag{4.10b}$$

So from (2.2), (2.3) and (3.1*a*, *b*)

$$(f_2)_{3x} + 6[w_{1,n-2}(k_1, k_2)]_x f_{2x} = k_2^3 f_2 \tag{4.11a}$$

$$(f_1)_{3x} + 6[w_{1,n-2}(k_1, k_2)]_x f_{1x} = k_1^3 f_1. \tag{4.11b}$$

Combining (4.6), (4.9) and (4.11) we have a solution of (4.6) with an $(n-1)$-soliton as potential in terms of the solutions of the same equation with an $(n-2)$-soliton as potential. We thus have a recursive scheme for obtaining the solution of the third-order scattering equation (2.3) with an n-soliton as potential.

We note that the eigenvalue k_1 in (4.6) is not a parameter of the $(n-1)$-soliton $w_{1,n-1}(k_1)$ which appears as the potential. If we require the solution of (4.6) with an eigenvalue k_i^3 which appears as a parameter in the $(n-1)$-soliton potential, that is a solution of

$$(g(k_1))_{3x} + 6(w_{1,n-1}(k_1))_x(g(k_1))_x = k_i^3 g(k_1) \qquad i = 2, 3, \ldots, n \tag{4.12}$$

then we consider one solution f of

$$\phi_{3x} + 6[w_{1,n-2}(k_1, k_i)]_x\phi_x = k_i^3\phi \tag{4.13}$$

such that

$$f_x/f = w_{1,n-1}(k_1) - w_{1,n-2}(k_1, k_i) \tag{4.14}$$

and another solution $\bar{f}$ of (4.13) which does not depend linearly on f. Then

$$g(k_1) = D^{-1}(f_x\bar{f} - f\bar{f}_x)/f \tag{4.15}$$

is a solution of (4.12).

In § 5 we will show that the eigenfunctions of the sixth-order recursion operator $T(w_{1,n})$ (see (5.6) for the definition) of the CDGSKE with eigenvalue $\lambda = -27k_1^6$ can be written in terms of the solutions of (4.6).

5. Eigenfunctions of the recursion operator for soliton solutions

Consider equation (4.5). Let $y(x, t)$, $h(x, t)$ and $z(x, t)$ be infinitesimal changes in $w_{1,n-1}(k_1)$, $g(k_1)$ and $w_{1,n}$ respectively. That is

$$\begin{aligned} &w_{1,n-1}(k_1) \to w_{1,n-1}(k_1)+\varepsilon y \\ &g(k_1) \to g(k_1)+\varepsilon h \\ &w_{1,n} \to w_{1,n}+\varepsilon z. \end{aligned} \tag{5.1}$$

Substituting (5.1) in (4.5) and equating terms linear in ε we obtain

$$z(x, t) = y(x, t) + (h/g(k_1))_x. \tag{5.2}$$

For $y(x, t) = 0$ we obtain

$$z(x, t) = (h/g(k_1))_x. \tag{5.3}$$

We claim that $(h/g(k_1))_x$ is an eigenfunction of the sixth-order recursion operator (Fuchssteiner and Oevel 1982) with eigenvalue $= -27k_1^6$. We show this in two steps. First the RHS of (5.3) is shown to satisfy (2.5) with $\bar{w} = w_{1,n}$ and $w = w_{1,n-1}(k_1)$, that is $z = (h/g(k_1))_x$ satisfies

$$z_{xx} + 3(w_{1,n} - w_{1,n-1}(k_1))z_x + 3[(w_{1,n} - w_{1,n-1}(k_1))^2 + (w_{1,n} + w_{1,n-1}(k_1))_x]z = 0. \tag{5.4}$$

It is then shown that any solution of (5.4) is an eigenfunction of the recursion operator $T(w_{1,n})$ with an eigenvalue $= -27k_1^6$. These proofs are direct and are sketched below. Since $g(k_1)$ satisfies the linear equation (4.6) and $y = 0$, $h(x, t)$ also satisfies (4.6), that is

$$h_{3x} + 6(w_{1,n-1}(k_1))_x h_x = k_1^3 h. \tag{5.5}$$

Evaluate z_x and z_{xx} from (5.3). We eliminate $w_{1,n}$ using (4.5). Substitute for h_{3x} and $(g(k_1))_{3x}$ using (4.6) and (5.5) and the result follows.

The sixth-order recursion operators for the CDGSKE (Fuchssteiner and Oevel 1982) can be written in the factorised form as (Aiyer 1984b)

$$\begin{aligned} T(w_{1,n}) = {} & D^{-1}[D^2 + 24(w_{1,n})_x + 12(w_{1,n})_{xx}D^{-1}] \\ & \times [D^2 + 6(w_{1,n})_x]D[D^2 + 6(w_{1,n})_x]. \end{aligned} \tag{5.6}$$

This generates the IT about $w_{1,n}$. We have to show that

$$T(w_{1,n})\{z\} = -27k_1^6 z \tag{5.7}$$

where $z(x, t)$ satisfies (5.4). The proof is direct. Equation (5.4) and the ABT

$$\begin{aligned} &\{w_{1,n} - w_{1,n-1}(k_1)\}_{xx} + \{w_{1,n} - w_{1,n-1}(k_1)\}^3 \\ &\quad + 3\{w_{1,n} - w_{1,n-1}(k_1)\}\{w_{1,n} + w_{1,n-1}(k_1)\}_x = k_1^3 \end{aligned} \tag{5.8}$$

have to be used repeatedly to reduce the higher derivatives of z and $w_{1,n}$. The only point which presents some difficulty is the evaluation of

$$D^{-1}[D^2 + 6(w_{1,n})_x]D[D^2 + 6(w_{1,n})_x]z. \tag{5.9}$$

This is evaluated in appendix 1.

The rest of the calculation is direct and one finally obtains (5.7). We have thus succeeded in obtaining the eigenfunctions of the sixth-order recursion operator $T(w_{1,n})$ in terms of the solutions of the linear equation (4.6) or (5.5). Furthermore, a method to explicitly construct the solution of this linear equation recursively has been given in § 4.

Now (5.5) has three linearly independent solutions which we denote by $h_i(x, t)$, $i = 1, 2, 3$. This might lead one to conclude that there are three independent eigenfunctions of the recursion operator belonging to the eigenvalue $-27k_1^6$. However only two of the three functions

$$(h_i/g(k_1))_x \qquad i = 1, 2, 3 \tag{5.10}$$

are linearly independent because $g(k_1)$ and h are solutions of the same linear equation and

$$g(k_1) = \sum_{i=1}^{3} \alpha_i h_i \tag{5.11}$$

for some constants α_i. Hence

$$\sum_{i=1}^{3} \alpha_i (h_i/g(k_1))_x = 0. \tag{5.12}$$

For each of the n parameters k_i, $i = 1$ to n, of the n-soliton we have two independent eigenfunctions giving $2n$ discrete eigenfunctions of the recursion operator $T(w_{1,n})$ which are also the IT about the n-soliton. In contrast, about the n-soliton of the KdVE one has n discrete IT. The extra factor of 2 arises from the two independent sets of IT for the CDGSKE.

6. 1-soliton solutions and their discrete IT

In this section we discuss the behaviour of the 1-soliton solution and some aspects of the two discrete IT about the 1-soliton.

By a 1-soliton w_1 we mean a solution $\bar{w} = w_1$ of (2.1) with $w = 0$. The equation for $w_1(x, t)$ is

$$(w_1)_{xx} + 3w_1 w_{1x} + w_1^3 = k_1^3. \tag{6.1}$$

Following Dodd and Gibbon (1977) (6.1) reduces to

$$\psi_{3x} = k_1^3 \psi \tag{6.2}$$

with

$$w_1 = \psi_x / \psi. \tag{6.3}$$

A general solution of (6.2) is

$$\psi = A_1 \exp(k_1 x) + A_2 \exp(\theta k_1 x) + A_3 \exp(\theta^* k_1 x) \tag{6.4}$$

where θ and $\theta^* = (-1 \pm i\sqrt{3})/2$ are the cube roots of unity.

We now come to the time dependence of $w_1(x, t)$. Being a 1-soliton it might be natural to assume that

$$w_1(x, t) \sim w_1(x - \alpha t). \tag{6.5}$$

However this is not true for every solution $w_1(x, t)$ when ψ has the general form (6.4). We found that $w_1(x, t)$ given by

$$w_1(x, t) = f_x(x, t)/f(x, t) \tag{6.6}$$

where

$$f(x,t)=[A_1\exp-\mathrm{i}\sqrt{3}(\beta x+at)+A_2\exp(\beta x-at)$$
$$+A_3\exp-(\beta x-at)]\exp(\mathrm{i}\beta/\sqrt{3})x \tag{6.7}$$

satisfies the CDGSKE (1.3) where a can be determined by requiring that (6.6) satisfies (1.3). The time variation of the oscillating part is different from the time variation of the other terms. The factor $\exp(\mathrm{i}\beta/\sqrt{3})x$ is of no consequence as it cancels when $w_1(x,t)$ is evaluated from (6.6). Here

$$\beta=(\mathrm{i}\sqrt{3}/2)k_1 \qquad \text{and} \qquad a=16\beta^4. \tag{6.8}$$

For $A_1=0$, $w_1(x,t)$ reduces to the result obtained by Dodd and Gibbon, namely ($A_2=A_3=1$, say)

$$w_1(x,t)=\beta[\tanh(\beta x-at)+\mathrm{i}/\sqrt{3}]. \tag{6.9}$$

The time dependence of $w_1(x,t)$ given by (6.6) and (6.7) can be viewed in the following way. In the 1-soliton manifold defined by (6.1) the CDGSKE (1.3) reduces to

$$w_{1t}-9k_1^3(w_{1xx}+2w_1w_{1x})=0 \tag{6.10}$$

that is

$$(w_1)_{5x}+3w_{1x}(w_1)_{3x}+60(w_{1x})^3 \tag{6.11}$$

reduces to

$$-9k_1^3(w_{1xx}+2w_1w_{1x}) \tag{6.12}$$

if one uses (6.1). (6.11) does not reduce to

$$\text{constant}\times w_{1x} \tag{6.13}$$

so that w_{1t} is not proportional to w_{1x} and this is also the case for the time dependence given by (6.6) and (6.7).

If however one considers the 1-soliton solution given by (6.9) then

$$w_{1t}=\text{constant}\times w_{1x} \tag{6.14}$$

but in this case

$$(w_1)_{xx}+2w_1w_{1x}=\text{constant}\times w_{1x} \tag{6.15}$$

as can be verified directly.

Let us now consider the IT about the 1-soliton arising from the trivial IT about the zero solution of the CDGSKE. This can be obtained by putting $\bar{w}=w_1$ and $w=0$ in (2.5) or directly from (6.1) with $w_1\to w_1+\varepsilon z$. The equation for the IT $z(x,t)$ about $w_1(x,t)$ is

$$z_{xx}+3w_1z_x+3(w_1^2+w_{1x})z=0. \tag{6.16}$$

It can be verified directly using (6.1) that $z_1=w_{1x}$ and $z_2=(w_{1x}+w_1^2)_x$ are solutions of (6.16) which is just the reduction of (6.11) or w_{1t} in the 1-soliton manifold. Therefore for a general solution $w_1(x,t)$ given by (6.6) and (6.7) the two solutions z_1 and z_2 of (6.16) are the two independent IT. However for $w_1(x,t)$ given by (6.9), we have seen that z_1 is proportional to z_2. But there does exist apart from z_1 another IT about $w_1(x,t)$ given by (6.9). This can be easily obtained from (6.16) using the Wronskian. This solution is

$$z_2=[\operatorname{sech}(\beta x-at)\exp(-\mathrm{i}\sqrt{3})(\beta x+at)]_x. \tag{6.17}$$

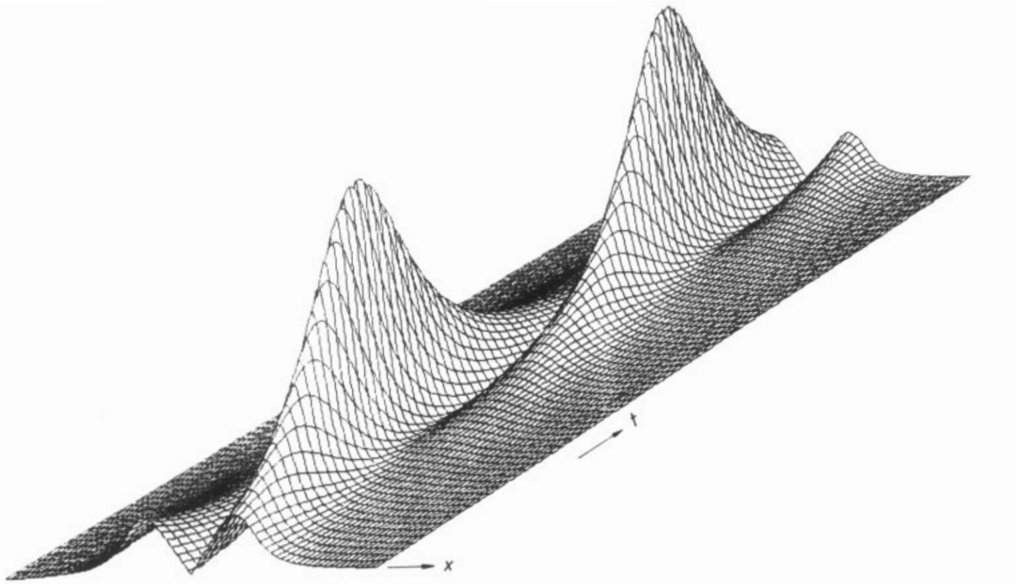

Figure 1. 1-soliton $|U_1(x, t)|$ for the case when no singularity develops. $A_1 = A_2 = A_3 = 1$; x range $= -5$–5; t range $= 0$–2 (periods).

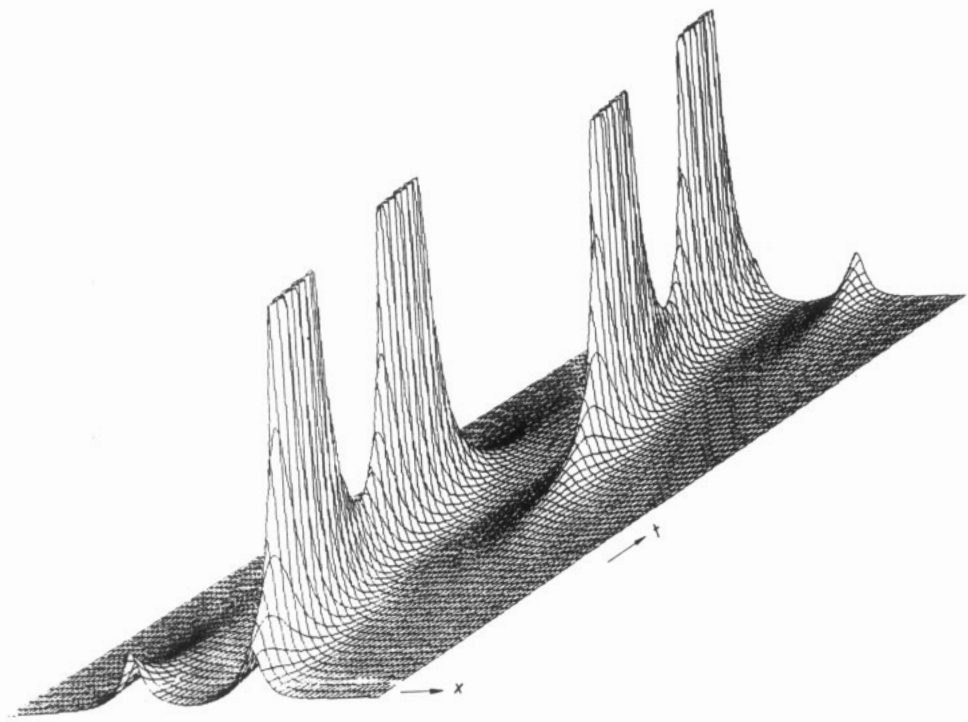

Figure 2. $|U_1(x, t)|$ for the case when an initially non-singular pulse develops singularity at two instants of time. $A_1 = 1$, $A_2 = A_3 = 0.3$; x range $= -5$–5; t range $= 0$–2 (periods).

This IT cannot be written in terms of w_1 and its partial derivatives as could be done for the general case. Also the time dependences of the sech term and the oscillatory term are different.

If one considers the seventh-order equation belonging to the CDGSK hierarchy, that is

$$w_t + T(w)\{w_x\} = 0 \tag{6.18}$$

where the recursion operator $T(w)$ is given by (5.6) with $w_{1,n} = w$ then in the 1-soliton manifold $T(w_1)\{w_{1x}\}$ reduces to $-27k_1^6 w_{1x}$ so that (6.18) reduces to

$$w_{1t} - 27k_1^6 w_{1x} = 0. \tag{6.19}$$

The 1-soliton solution would again be given by (6.6) but with the important difference that the time dependence in (6.7) is the same, namely $(\beta x - at)$, in all the terms. This is an interesting difference in the behaviour of the 1-soliton solution (and so also for the higher solitons) of the NLEE belonging to the same hierarchy.

We now discuss some of the novel features of the behaviour of the 1-soliton $u_1(x, t) = w_{1x}(x, t)$. We have observed earlier that with $A_1 = 0$ and $A_2 = A_3 = 1$ we obtain the usual soliton solution.

$$u_1(x, t) \sim \operatorname{sech}^2(\beta x - at). \tag{6.20}$$

A singular solution

$$u_1(x, t) \sim \operatorname{cosech}^2(\beta x - at) \tag{6.21}$$

is obtained with $A_1 = 0$, $A_2 = -A_3 = 1$.

If $A_1 \neq 0$ then certain novel features appear. Let us take $A_1 = 1$ without any loss of generality. In this case f given by (6.7) is complex and $|u_1(x, t)|$ tends to infinity only when both the real and imaginary parts of $f(x, t)$ vanish. An analysis of the values of (x, t) when this can happen leads to the following two possibilities.

(a) A solution non-singular to start with continues to propagate as a non-singular solution.

(b) A solution non-singular to start with becomes singular at one or at most two instants of time. That is, a non-singular solution becomes singular as it evolves in time and again becomes non-singular.

Whether $|u_1(x, t)|$ behaves according to (a) or (b) depends upon the values of A_1, A_2 and A_3. For example we have plotted the evolution of $|u_1(x, t)|$ for $A_1 = A_2 = A_3 = 1$ (see figure 1) when it behaves as in (a) and $A_1 = 1$, $A_2 = A_3 = 0.3$ (see figure 2) when it behaves as in (b). The behaviour described in (b) is to be contrasted with that given by (6.21) where the solution has a singularity for all t.

Another interesting feature of the solution when $A_1 \neq 0$ is that $|u_1(x, t)|$ starts as a symmetrical double-peaked curve. One of the peaks rises at the cost of the other, reaches a maximum and then falls while the other begins to grow. In a full period the symmetrical shape is regained. This behaviour is reminiscent of the solution of the double SGE (Bullough *et al* 1980) where the peaks wobble. A possible reason for this similarity to the solution of the DSGE is given in § 7.

7. The SG-like equation from the CDGSKE

The scheme for deriving the SG-like equation from the CDGSKE is the following. We will first consider how to obtain the SGE from the KdVE.

Using the Miura transform one obtains the modified KdVE from the KdVE. One can then go to the potential form of the MKdVE. Using the inverse of the recursion operator for this potential form one obtains the usual SGE (Fuchssteiner 1979, Fokas and Fuchssteiner 1981a, b, Aiyer 1983a, b). We follow a similar procedure for the CDGSKE.

We start from the CDGSKE for $u(x, t)$ given by (1.1), namely

$$-u_t = u_{5x} + 30uu_{3x} + 30u_xu_{xx} + 180u^2u_x. \tag{7.1}$$

Let us transform the dependent variable u in (7.1) by the Miura transform to a new variable $\mathscr{V}(x, t)$

$$6u = \mathrm{i}\mathscr{V}_x + \mathscr{V}^2. \tag{7.2}$$

Then $\mathscr{V}(x, t)$ satisfies the modified SK (MSK) equation (Fordy and Gibbons 1980)

$$-\mathscr{V}_t = \mathscr{V}_{5x} - 5\mathrm{i}\mathscr{V}_x\mathscr{V}_{3x} + 5\mathscr{V}^2\mathscr{V}_{3x} - 5\mathrm{i}\mathscr{V}_{xx}^2 + 20\mathscr{V}\mathscr{V}_x\mathscr{V}_{xx} + 5\mathscr{V}_x^3 + 5\mathscr{V}^4\mathscr{V}_x.$$

Using the proper transformation laws (Fokas and Fuchssteiner 1981a, b) one obtains the recursion operator T_{MSK} for the MSK equation with $T_{\mathrm{SK}}(w)$ given by (5.6)

$$T_{\mathrm{MSK}}(\mathscr{V}) = (\mathrm{i}D + 2\mathscr{V})^{-1} D T_{\mathrm{SK}}(w) D^{-1}(\mathrm{i}D + 2\mathscr{V}). \tag{7.3}$$

To obtain the potential MSK equation we put

$$\mathscr{V} = \phi_x \tag{7.4}$$

and find the NLEE for $\phi(x, t)$ from (7.1), (7.2) and (7.4)

$$-\phi_t = \phi_{5x} + 5\phi_x^2\phi_{3x} - 5\mathrm{i}\phi_{xx}\phi_{3x} + 5\phi_x\phi_{xx}^2 + \phi_x^5 \equiv \bar{z} \tag{7.5}$$

which admits the recursion operator $T_{\mathrm{PMSK}}(\phi)$

$$T_{\mathrm{PMSK}}(\phi) = D^{-1} T_{\mathrm{MSK}}(\phi) D. \tag{7.6}$$

To obtain the equation corresponding to the SGE for CDGSKE we apply $T_{\mathrm{PMSK}}^{-1}(\phi)$ to $\bar{z}$, the RHS of (7.5). The derivation of $T_{\mathrm{PMSK}}(\phi)$ is given in appendix 2. It is given by

$$\begin{aligned} T_{\mathrm{PMSK}}^{-1}(\phi) = {} & D^{-1}(D - 2\mathrm{i}\phi_x)^{-1} D (D + \mathrm{i}\phi_x)^{-1}(D - \mathrm{i}\phi_x)^{-1} D^{-1} \\ & \times (D + \mathrm{i}\phi_x)^{-1}(D - \mathrm{i}\phi_x)^{-1} D (D + 2\mathrm{i}\phi_x)^{-1}. \end{aligned} \tag{7.7}$$

Using

$$(D + \alpha\phi_x)^{-1} = \exp(-\alpha\phi) \int_{-\infty}^{x} \mathrm{d}x_1 \exp(\alpha\phi) \tag{7.8}$$

we obtain using (7.5)

$$T_{\mathrm{PMSK}}^{-1}(\phi)\{\bar{z}\} = D^{-1}(\exp(-\mathrm{i}\phi) - \exp(2\mathrm{i}\phi)) \tag{7.9}$$

where the boundary condition $\phi \to \pm 2n\pi$ as $x \to -\infty$ is used. The SG-like equation for the CDGSKE is therefore

$$\phi_t = T_{\mathrm{PMSK}}^{-1}(\phi)\{\bar{z}\}. \tag{7.10}$$

Simplifying the RHS of (7.9) we obtain from (7.9) and (7.10)

$$\phi_{xt} = \sin\phi + \exp(3\mathrm{i}\phi/2)\sin(\phi/2) \tag{7.11}$$

which resembles the double SGE (Bullough *et al* 1980).

8. Conclusion

The ABT for the CDGSKE is of second order. By a single iteration we have obtained a first-order equation for the n-soliton in terms of $(n-1)$- and $(n-2)$-solitons. Using this we have constructed recursively the solution of the third-order isospectral equation with an n-soliton as the potential in terms of the solution of the same equation with an $(n-1)$-soliton as potential. The discrete eigenfunctions of the sixth-order recursion operator $T(w_{1,n})$ where $w_{1,n}$ is an n-soliton is then found in terms of the solutions of the isospectral equation. Some details about the behaviour of 1-solitons and their IT are studied. Finally that equation for the CDGSKE which is equivalent to the SGE for the KdVE is derived.

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Appendix 1

Recalling equation (5.9), we have

$$D^{-1}Y\equiv D^{-1}[D^2+6(w_{1,n})_x]D[D^2+6(w_{1,n})_x]z.$$

We present the method of evaluating (5.9). It can be shown directly that

$$\begin{aligned}Y=9z_x\{&2[w_{1,n}-w_{1,n-1}(k_1)]^4-2[w_{1,n}-w_{1,n-1}(k_1)]^2\\&\times[w_{1,n}-w_{1,n-1}(k_1)]_x-k_1^3[w_{1,n}-w_{1,n-1}(k_1)]\}\\&+9z\{2[w_{1,n}-w_{1,n-1}(k_1)]^5-4[w_{1,n}-w_{1,n-1}(k_1)]^3\\&\times[w_{1,n}-w_{1,n-1}(k_1)]_x+k_1^3[w_{1,n}-w_{1,n-1}(k_1)]^2\\&+2[w_{1,n}-w_{1,n-1}(k_1)][(w_{1,n}-w_{1,n-1}(k_1))_x]^2\\&-k_1^3[w_{1,n}-w_{1,n-1}(k_1)]_x\}.\end{aligned}\tag{A1.1}$$

Integrating by parts, we have

$$\begin{aligned}D^{-1}Y=9z\{&2[w_{1,n}-w_{1,n-1}(k_1)]^4-2[w_{1,n}-w_{1,n-1}(k_1)]^2\\&\times[w_{1,n}-w_{1,n-1}(k_1)]_x-k_1^3[w_{1,n}-w_{1,n-1}(k_1)]\}\\&+27\int_{-\infty}^{x}\{2[w_{1,n}-w_{1,n-1}(k_1)]^3[-3w_{1,n}+w_{1,n-1}(k_1)]_x\\&+2[w_{1,n}-w_{1,n-1}(k_1)][(w_{1,n}-w_{1,n-1}(k_1))_x]^2\\&+k_1^3[w_{1,n}-w_{1,n-1}(k_1)]^2\}z\,\mathrm{d}x.\end{aligned}\tag{A1.2}$$

The problem is to evaluate the integral in (A1.2). For this we consider the integral

$$\begin{aligned}&\int 3[w_{1,n}-w_{1,n-1}(k_1)]^5z\,\mathrm{d}x\\&\qquad=\int[w_{1,n}-w_{1,n-1}(k_1)]^3 3[w_{1,n}-w_{1,n-1}(k_1)]^2z\,\mathrm{d}x.\end{aligned}\tag{A1.3}$$

Substituting for $3[w_{1,n}-w_{1,n-1}(k_1)]^2z$ from (5.4) and integrating by parts, we obtain

$$3\int[w_{1,n}-w_{1,n-1}(k_1)]^5z\,\mathrm{d}x$$
$$=-[w_{1,n}-w_{1,n-1}(k_1)]^3z_x+3[w_{1,n}-w_{1,n-1}(k_1)]^2[w_{1,n}-w_{1,n-1}(k_1)]_xz$$
$$-3[w_{1,n}-w_{1,n-1}(k_1)]^4z+\int 6z\{[w_{1,n}-w_{1,n-1}(k_1)]^3$$
$$\times[3w_{1,n}-w_{1,n-1}(k_1)]_x-[w_{1,n}-w_{1,n-1}(k_1)][(w_{1,n}-w_{1,n-1}(k_1))_x]^2$$
$$-\tfrac{1}{2}k_1^3[w_{1,n}-w_{1,n-1}(k_1)]^2+\tfrac{1}{2}[w_{1,n}-w_{1,n-1}(k_1)]^5\}\,\mathrm{d}x. \qquad \text{(A1.4)}$$

Cancelling $\int 3[w_{1,n}-w_{1,n-1}(k_1)]^5z\,\mathrm{d}x$ we obtain the desired integral. Finally

$$D^{-1}Y=-9[w_{1,n}-w_{1,n-1}(k_1)]^3z_x+9z\{-[w_{1,n}-w_{1,n-1}(k_1)]^4+[w_{1,n}-w_{1,n-1}(k_1)]^2$$
$$\times[w_{1,n}-w_{1,n-1}(k_1)]_x-k_1^3[w_{1,n}-w_{1,n-1}(k_1)]\}. \qquad \text{(A1.5)}$$

Appendix 2

From (7.3), (7.4) and (7.6)

$$T_{\mathrm{PMSK}}(\phi)=D^{-1}(\mathrm{i}D+2\phi_x)^{-1}DT_{\mathrm{SK}}(w)D^{-1}(\mathrm{i}D+2\phi_x). \qquad \text{(A2.1)}$$

The problem of finding $T^{-1}_{\mathrm{PMSK}}(\phi)$ is therefore reduced to finding $T^{-1}_{\mathrm{SK}}(w)$ where $T_{\mathrm{SK}}(w)$ is given by (5.6). $T^{-1}_{\mathrm{SK}}(w)$ is obtained below.

With

$$\mathcal{V}=\mathrm{i}\psi_x/\psi \qquad \text{(A2.2)}$$

and using (7.2) we obtain

$$6u=6w_x=-\psi_{xx}/\psi. \qquad \text{(A2.3)}$$

Substituting (A2.3) in (5.6) we can further factorise (5.6) to give

$$T_{\mathrm{SK}}(w)=D^{-1}(D+2\psi_x/\psi)D(D-2\psi_x/\psi)D^{-1}$$
$$\times(D+\psi_x/\psi)(D-\psi_x/\psi)D(D+\psi_x/\psi)(D-\psi_x/\psi). \qquad \text{(A2.4)}$$

Since $T_{\mathrm{SK}}(w)$ is the product of first-order differential operators its inverse can easily be found.

Combining (7.4), (A2.2) and (A2.1) we obtain

$$T_{\mathrm{PMSK}}(\phi)=(D+2\mathrm{i}\phi_x)D^{-1}(D-\mathrm{i}\phi_x)(D+\mathrm{i}\phi_x)$$
$$\times D(D-\mathrm{i}\phi_x)(D+\mathrm{i}\phi_x)D^{-1}(D-2\mathrm{i}\phi_x)D \qquad \text{(A2.5)}$$

whose inverse is given by (7.7) and (7.8).

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